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COMMENT

# On the solution of the steady state equation for two-species annihilation

S Simons

School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, UK

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**Abstract.** It is shown that for a particular physical situation the steady state equation describing two-species annihilation possesses a simple analytic solution.

A recent paper by Ben-Naim and Redner (1992) was concerned with solving the equations describing the steady-state two-species annihilation process $A+B \rightarrow 0$, and for the various physical situations in which they were interested, they obtained approximate solutions. The fact that they were unable to obtain exact solutions is not surprising since the relevant differential equations are coupled, second order and nonlinear. The purpose of the present communication is to point out that there exists a particular well defined physical situation for which the equations do possess a simple analytic solution.

We consider diffusion of the two species in the $x$ direction and let $C_A(x)$ and $C_B(x)$ be the concentration of A and B respectively. Then in the steady state $C_A$ and $C_B$ satisfy

$$D_A \, d^2C_A/dx^2 = kC_AC_B \tag{1a}$$

$$D_B \, d^2C_B/dx^2 = kC_AC_B \tag{1b}$$

where $D_A$, $D_B$ are the respective diffusion coefficients for A and B, and $k$ is the reaction constant. We now define $c_A = (k/D_B)C_A$, $c_B = (k/D_A)C_B$ and obtain

$$d^2c_A/dx^2 = c_Ac_B \tag{2a}$$

$$d^2c_B/dx^2 = c_Ac_B. \tag{2b}$$

We suppose that the source of both species is at $x=0$ and that diffusion occurs within the interval $0 \leqslant x \leqslant \infty$. At $x=0$ we consider two types of boundary condition: either that $c_A$ and $c_B$ are each maintained constant at the same value $c_0$, or that there is the same input flux $J_0$ for each of the two species. These conditions correspond respectively to

$$c_A(0) = c_B(0) = c_0 \tag{3a}$$

and

$$c'_A(0) = c'_B(0) = -kJ_0/D_AD_B. \tag{3b}$$

At $x=\infty$ there will be complete mutual annihilation of the two species corresponding to the infinite time required to diffuse there and the second boundary condition therefore takes the form

$$c_A(\infty) = c_B(\infty) = 0. \tag{4}$$

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To solve these equations we let $c_-(x) = c_A(x) - c_B(x)$, when $c_-(x)$ satisfies $d^2c_-/dx^2 = 0$ together with boundary conditions $c_-(\infty) = 0$ and either $c_-(0) = 0$ or $c'_-(0) = 0$. In both cases the solution is $c_-(x) = 0$ corresponding to $c_A(x) = c_B(x)$, and allowing equations (2) to be expressed as

$$d^2c/dx^2 = [c(x)]^2 \tag{5}$$

($c_A = c_B = c$) with boundary conditions (3) and (4). For boundary conditions (3*a*) and (4), equation (5) has the simple solution

$$c(x) = \frac{c_0}{[1 + (c_0/6)^{1/2}x]^2} \tag{6a}$$

with $J(x)/J(0) = [1 + (c_0/6)^{1/2}x]^{-3}$ and $J(0) = (2/\sqrt{6})D_A D_B c_0^{3/2}/k$, being thus proportional to $c_0^{3/2}$. For the boundary conditions (3*b*) and (4) the solution of equation (5) is

$$c(x) = \frac{6}{[(12D_A D_B/kJ_0)^{1/3} + x]^2} \tag{6b}$$

and

$$\frac{J(x)}{J_0} = \left[1 + \left(\frac{kJ_0}{12D_A D_B}\right)^{1/3} x\right]^{-3}.$$

## References

Ben-Naim E and Redner S 1992 *J. Phys. A: Math. Gen.* **25** L575